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A characterization of the Schechter essential spectrum on Banach spaces and applications

Aref Jeribi

Département de Mathématiques, Faculté des Sciences de Gabès, Route de Médenine, 6029 Gabès, Tunisia

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Abstract

In a recent article by the author (C. R. Acad. Sci. Paris Sér. I 331 (2000) 525–530; Boll. Un. Mat. Ital. (2002), to appear) the Schechter spectrum of closed, densely defined linear operators has been characterized on spaces, which possess the Dunford–Pettis property or which are isomorphic to one of the spaces $L_p(\Omega)$, $p > 1$. The purpose of the present work is to extend this analysis to the case of Banach spaces. Further we apply the obtained results to investigate the Schechter essential spectrum of one-dimensional transport equations with different boundary conditions. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

Let A be a closed, densely defined linear operator on a Banach space X, and let $\sigma(A)$ (respectively, $\rho(A)$) denote the spectrum (respectively, the resolvent set) of A. We denote by $\mathcal{C}(X)$ (respectively, $\mathcal{L}(X)$) the set of all closed, densely defined linear operators (respectively, the set of all bounded linear operators) on X to itself and $\mathcal{K}(X)$ the ideal of compact operators of $\mathcal{L}(X)$.

Definition 1.1. Let $A \in \mathcal{C}(X)$. We define the essential spectrum of the operator A by

E-mail address: aref.jeribi@fsg.rnu.tn.

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$$\sigma_{\text{ess}}(A) = \bigcap_{C \in \mathcal{K}(X)} \sigma(A + C).$$

It is well known that if A is a self-adjoint operator on a Hilbert space, the essential spectrum of A is the set of limit points of the spectrum of A (with eigenvalues counted according to their multiplicities), i.e., all points of the spectrum except isolated eigenvalues of finite multiplicity (see, for example, [45,46]).

There are many ways to define the essential spectrum of a closed, densely defined linear operator on a Banach space. Hence several definitions of the essential spectrum may be found in the literature (see, for example, [11,37] or the comments in [39, Chapter 11, p. 283]) which coincide for self-adjoint operators on Hilbert spaces. Throughout this paper we are concerned with the so-called Schechter essential spectrum.

Definition 1.2. An operator $A \in \mathcal{L}(X)$ is called strictly singular if A is not an isomorphism when restricted to any infinite-dimensional subspace of X.

The concept of strictly singular operators was introduced in the pioneering paper by Kato [19] as a generalization of the notion of compact operators. The class of strictly singular operators has been extensively studied in the late 60's (see, for example, [6,7,32,36] and references therein). For our own use, let us recall the following three facts. The set of all strictly singular operators on X, $\mathcal{S}(X)$, forms a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$; if X is a Hilbert space then $\mathcal{K}(X) = \mathcal{S}(X)$ and the class of weakly compact operators on L_1-spaces is nothing else but the family of strictly singular operators on L_1-spaces (see [36, Theorem 1]).

In 1996 and 1998, motivated by a problem concerning the spectrum of the transport operator posed in [21], Latrach and Jeribi [16,27,29] obtained the following result:

Theorem 1.1 [29, Theorem 3.2]. *Let (Ω, Σ, μ) be an arbitrary positive measure space. If A is a closed densely defined linear operator on $L_p(\Omega)$ $(1 \leqslant p < \infty)$ then*

$$\sigma_{\text{ess}}(A) = \bigcap_{S \in \mathcal{S}(L_p(\Omega))} \sigma(A + S),$$

where $\mathcal{S}(L_p(\Omega))$ stands for the ideal of strictly singular operators on $L_p(\Omega)$.

Definition 1.3. A Banach space X is said to have the Dunford–Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \to Y$ takes weakly compact sets in X into norm compact sets of Y.

It is well known that any L_1 space has the property DP [4]. Also, if Ω is a compact Hausdorff space, $C(\Omega)$ has the property DP [10]. For further examples we refer to [3] or [5, pp. 494, 497, 508, and 511]. Note that the property DP is not preserved under conjugation. However, if X is a Banach space whose dual has the property DP then X has the property DP (see, e.g., [10]). For more information we refer to the paper by Diestel [3] which contains a survey and exposition of the Dunford–Pettis property and related topics.

Definition 1.4. An operator $A \in \mathcal{L}(X)$ is said to be weakly compact if $A(B)$ is relatively weakly compact for every bounded subset $B \subset X$.

The family of weakly compact operators on X, $\mathcal{W}(X)$, is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (cf. [5,6]). Note also that if $X = L_1(\Omega, \Sigma, d\mu)$, where $(\Omega, \Sigma, d\mu)$ is a positive measure space, or $X = C(K)$ with K is a compact Hausdorff space then $\mathcal{W}(X) = \mathcal{S}(X)$ (cf. [36]).

In 1999 Latrach [25] gives an extension of the Theorem 1.1 to general Banach spaces which possess the Dunford–Pettis property in terms of weakly compact operators and obtained the following results:

Theorem 1.2 [25, Theorem 3.2]. *Let $A \in \mathcal{C}(X)$. If X has the Dunford–Pettis property, then*

$$\sigma_{\mathrm{ess}}(A) = \bigcap_{F \in \mathcal{W}(X)} \sigma(A + F).$$

Let $A \in \mathcal{C}(X)$; we suppose that the Schechter essential spectrum of A is known. Let us perturb the operator A with the bounded operator K, i.e., $A + K$. What will the Schechter essential spectrum of the operator $A + K$ be? If K is a compact operator on Banach spaces then $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$ (see Definition 1.1). If K is a strictly singular on L_p-spaces then $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$ (see Theorem 1.1). If K is a weakly compact on Banach spaces which possess the Dunford–Pettis property then $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$ (see Theorem 1.2). But in practice, the perturbed operator K is neither strictly singular nor weakly compact (see Section 3). So, it is natural to ask what are the conditions that we must impose on K such that $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$. For this, Jeribi [14,15] give a positive answers of this open question in particular space. In fact:

Theorem 1.3. *Let X be a Banach space and let $A \in \mathcal{C}(X)$.*

If X has the Dunford–Pettis property then

$$\sigma_{\mathrm{ess}}(A) = \bigcap_{C \in \mathcal{G}_A^F(X)} \sigma(A + C),$$

where $\mathcal{G}_A^F(X) = \{K \in \mathcal{L}(X)$ such that $(\lambda - A)^{-1}K \in \mathcal{W}(X)$ for some $\lambda \in \rho(A)\}$.

If X is isomorphic to one of the spaces $L_p(\Omega)$, $p > 1$, then

$$\sigma_{\mathrm{ess}}(A) = \bigcap_{C \in \mathcal{G}_A^S(X)} \sigma(A + C),$$

where $\mathcal{G}_A^S(X) = \{K \in \mathcal{L}(X)$ such that $(\lambda - A)^{-1}K \in \mathcal{S}(X)$ for some $\lambda \in \rho(A)\}$.

This work is inspired by [15] where Schechter essential spectrum of non-self-adjoint, closed, densely defined linear operators are discussed on Banach space which possess the Dunford–Pettis property or which isomorphic to one of the spaces $L_p(\Omega)$, $p > 1$. The analysis uses the concept of weakly compact operators and strictly singular operators which possess some nice properties on theses spaces (cf. [6,32]). The aim of this paper consists in providing a detailed analysis of Schechter essential spectrum of non-self-adjoint, closed, densely defined linear operators on Banach spaces and gives an extension of the result of the Theorem 1.3 to Banach spaces. More precisely, let $A \in \mathcal{C}(X)$ and let $\mathcal{I}(X)$ be an arbitrary two-sided ideal of $\mathcal{L}(X)$. If $\mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X)$, where $\mathcal{F}(X)$ denotes the set of Fredholm perturbations, then $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$ for all $K \in \mathcal{L}(X)$ such that $(\lambda - A)^{-1}K \in \mathcal{I}(X)$ or for all $K \in \mathcal{L}(X)$ such that $K(\lambda - A)^{-1} \in \mathcal{I}(X)$. Our results extend and improve many known ones in the literature.

In the last section we will apply the results described above to investigate the Schechter essential spectrum of the following integro-differential operator:

$$\begin{aligned} A_H\psi(x,\xi) &= -\xi\frac{\partial\psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi) + \int_{-1}^{1}\kappa(x,\xi,\xi')\psi(x,\xi')\,d\xi' \\ &= T_H\psi + K\psi \end{aligned}$$

with general boundary conditions where $x \in [-a, a]$, $a > 0$, and $\xi \in [-1, 1]$. This operator describes the transport of particles (neutrons, photons, molecules of gas, etc.) in a plane parallel domain with a width of $2a$ mean free paths. The function $\psi(x, \xi)$ represents the number (or probability) density of gas particles having the position x and the direction cosine of propagation ξ. (The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x-direction). The functions $\sigma(.)$ and $\kappa(.,.,.)$ are called, respectively, the collision frequency and the scattering kernel. The boundary conditions are modeled by

$$\psi_{|\Gamma_-} = H\psi_{|\Gamma_+},$$

where Γ_- (respectively, Γ_+) is the incoming (respectively, outgoing) part of the phase space boundary, $\psi_{|\Gamma_-}$ (respectively, $\psi_{|\Gamma_+}$) is the restriction of ψ to Γ_- (respectively, Γ_+) and H is a linear bounded operator from a suitable function space on Γ_+ to a similar one on Γ_-.

In the classical neutron transport theory ($H = 0$), it is well known that

$$\sigma_{\text{ess}}(T_0+K)=\Big\{\lambda\in\mathbf{C}\text{ such that }\operatorname{Re}\lambda\leqslant-\liminf_{|\xi|\to 0}\sigma(\xi)\Big\}\quad\text{if } K=0. \tag{1.1}$$

If $K \neq 0$ and if some power of $(\lambda - T_0)^{-1}K$ is compact then it is well known that $\sigma(T_0 + K) \cap \{\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > -\liminf_{|\xi|\to 0}\sigma(\xi)\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities (see, for instance, [18,22] or [34]). On the other hand, under the above assumptions, the half plane $\{\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda \leqslant -\liminf_{|\xi|\to 0}\sigma(\xi)\}$ *may contain, a priori, some holes in the resolvent set of* $T_0 + K$. So (1.1) is not, a priori, true if $K \neq 0$ and some power of $(\lambda - T_0)^{-1}K$ is compact. These remarks remain valid if instead of K we consider the boundary operator H or both K and H. By taking advantage of the results of Section 2 and the compactness results obtained in [15], we are going to prove that (1.1) is, actually, true for general classes of boundary and collision operators H and K. More precisely, we give sufficient conditions on the collision operators K under which $\sigma_{\text{ess}}(T_H + K) = \sigma_{\text{ess}}(T_H)$ regardless of the boundary operator H.

Note that even though the spectral theory of transport operators is a classical theme in transport theory, generally, the analysis focuses on the point spectrum of these operators (see, for instance, [9,23,24,33,41,43] or [35]). In fact, the knowledge of the (peripheral) point spectrum permits to obtain a simple description of the time asymptotic behaviour ($t \to \infty$) of the solution of the associated Cauchy problem (cf. [9,22,43] or [34]).

The plan of the paper is as follows: The next section is devoted to the Schechter essential spectrum of closed, densely defined linear operators on Banach spaces. The main result of this section is Theorem 2.1. In Section 3 we apply the results obtained in Section 2 to investigate the Schechter essential spectrum of the one-dimensional transport operator with general boundary conditions (Theorem 3.1). We discuss briefly, by discussing the Schechter essential spectrum of transport operator, one in a bounded convex geometry (Theorem 3.2), another one in a slab geometry with vacuum boundary conditions (Theorem 3.3), another one in a slab geometry with generalized boundary conditions (Theorem 3.4) and the last one in a slab geometry with generalized periodic boundary conditions (Theorem 3.5).

2. The main result

Let X be a complex Banach space. By an operator A on X we mean a linear operator with domain $D(A) \subset X$ and range $R(A) \subset X$. For $A \in \mathcal{C}(X)$, we let $N(A)$ denote the null space of A, respectively. The nullity, $\alpha(A)$, of A is defined as the dimension $N(A)$, and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in X. The set of Fredholm operators is defined by

$$\begin{aligned}\Phi(X)=\big\{&A\in\mathcal{C}(X)\text{ such that }\alpha(A)<\infty,\\ &R(A)\text{ is closed in }X\text{ and }\beta(A)<\infty\big\},\end{aligned}$$

and the set of upper semi-Fredholm operators is defined by

$$\Phi_+(X) = \{A \in \mathcal{C}(X) \text{ such that } \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } X\}.$$

The Fredholm domain of A, Φ_A, is given by

$$\Phi_A := \{\lambda \in \mathbf{C} \text{ such that } \lambda - A \text{ is a Fredholm operator on } X\}.$$

Let X be a Banach space. If N is a closed subspace of X, we denote by π_N the quotient map $X \to X/N$. The codimension of N, codim(N), is defined to be the dimension of the vector space X/N.

Definition 2.1. Let X be a Banach space and $S \in \mathcal{L}(X)$. S is said to be strictly cosingular if there exists no closed subspace N of X with codim$(N) = \infty$ such that $\pi_N S : X \to X/N$ is surjective.

Let $\mathcal{CS}(X)$ denote the set of strictly cosingular operators on X. This class of operators was introduced by Pelczynski [36]. It forms a closed two-sided ideal of $\mathcal{L}(X)$ (cf. [42]).

Definition 2.2. Let X be a Banach space and $F \in \mathcal{L}(X)$. F is called a Fredholm perturbation if $U + F \in \Phi(X)$ whenever $U \in \Phi(X)$.

The set of Fredholm perturbations is denoted by $\mathcal{F}(X)$.

Remark 2.1. Let $\Phi^b(X)$ denote the set $\Phi(X) \cap \mathcal{L}(X)$. If in Definition 2.2 we replace $\Phi(X)$ by $\Phi^b(X)$, we obtain the set $\mathcal{F}^b(X)$. This class of operators was introduced and investigated in [7]. In particular, it is shown that $\mathcal{F}^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Definition 2.3. Let X be a Banach space and $R \in \mathcal{L}(X)$. R is said to be a Riesz operator if $\Phi_R = \mathbf{C} \setminus \{0\}$.

For further information on the family of Riesz operators we refer to [1,17] and references therein.

Remark 2.2. (a) The family of Riesz operators is not an ideal of $\mathcal{L}(X)$ (see [1]).
(b) In [38] it is proved that $\mathcal{F}^b(X)$ is the largest ideal of $\mathcal{L}(X)$ contained in the family of Riesz operators.

The following identity was established in [26, Lemma 2.3(ii)].

Lemma 2.1 [26]. *Let X be an arbitrary Banach space. Then*

$$\mathcal{F}(X) = \mathcal{F}^b(X).$$

An immediate consequence of this result is that $\mathcal{F}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Definition 2.4. Let X be a Banach space and $F \in \mathcal{L}(X)$. F is called a upper Fredholm perturbation if $U + F \in \Phi_+(X) \cap \mathcal{L}(X)$ whenever $U \in \Phi_+(X) \cap \mathcal{L}(X)$.

The set of upper semi-Fredholm perturbations is denoted by $\mathcal{F}_+^b(X)$. In [7] it is shown that $\mathcal{F}_+^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. In general, we have

$$\mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{F}_+^b(X) \subset \mathcal{F}(X),$$
$$\mathcal{K}(X) \subset C\mathcal{S}(X) \subset \mathcal{F}(X).$$

Let X be a fixed Banach space and let $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the condition

$$\text{(H1)} \quad \mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X).$$

Remark 2.3. It should be observed that if $\mathcal{I}(X)$ is a nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the condition $\mathcal{I}(X) \subset \mathcal{F}(X)$, then $\mathcal{F}_0(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X)$, where $\mathcal{F}_0(X)$ stands for the ideal of finite rank operators. This follows from Lemma 2.1 and [7, p. 70, Proposition 4].

We define the right spectrum of A by

$$\sigma_r(A) = \bigcap_{K \in \mathcal{G}_A(X)} \sigma(A + K),$$

where $\mathcal{G}_A(X) = \{K \in \mathcal{L}(X) \text{ such that } (\lambda - A)^{-1}K \in \mathcal{I}(X) \text{ for some } \lambda \in \rho(A)\}$. Similarly, we define the left spectrum of A by

$$\sigma_l(A) = \bigcap_{K \in \mathcal{D}_A(X)} \sigma(A + K),$$

where $\mathcal{D}_A(X) = \{K \in \mathcal{L}(X) \text{ such that } K(\lambda - A)^{-1} \in \mathcal{I}(X) \text{ for some } \lambda \in \rho(A)\}$.

The main result of this section is the following:

Theorem 2.1. *Let X be a Banach space, $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the hypothesis* (H1) *and let $A \in \mathcal{C}(X)$. Then*

$$\sigma_{\text{ess}}(A) = \sigma_r(A) = \sigma_l(A).$$

Remark 2.4. (a) Note that any subset $\mathcal{I}(X)$ of $\mathcal{L}(X)$ (not necessarily an ideal) satisfying the hypothesis (H1) may characterize the Schechter essential spectrum. Since $\mathcal{K}(X) \subset \mathcal{G}_A(X)$ and $\mathcal{K}(X) \subset \mathcal{D}_A(X)$, so $\mathcal{K}(X)$ is then the minimal subset of $\mathcal{L}(X)$ (in the sense of inclusion) for which Theorem 2.1 holds true. Hence Theorem 2.1 provides an improvement of the definition of $\sigma_{\text{ess}}(.)$ valid for a somewhat large variety of subsets of $\mathcal{L}(X)$. Also, it may be viewed as an extension of [14, Theorem 3.1], [16, Theorem 1], [15, Theorem 3.2], [12, Theorem 1], [13, Theorem 2.1] and [21, Theorem 3.4] to general Banach spaces.

(b) If X has the Dunford–Pettis property (respectively, is isomorphic to one of the spaces $L_p(\Omega, \Sigma, d\mu)$, $p > 1$, where (Ω, Σ, μ) is a positive measure space) and if we take $\mathcal{I}(X) = \mathcal{W}(X)$ (respectively, $\mathcal{I}(X) = \mathcal{S}(X)$) then we find again the Theorem 2.1 established in [13] or Theorem 1 established in [12]. Accordingly, Theorem 2.1 is a new characterization of the Schechter essential spectrum on these spaces. So, Theorem 2.1 may be regarded as an extension of [14, Theorem 1] and [15, Theorem 2.1] to general Banach spaces.

(c) Note that in applications (transport operators, operators arising in dynamic populations, etc.; see [15,28–30]), the operator B is, in general, a bounded perturbation of $A \in \mathcal{C}(L_p)$ by an integral operator on L_p-spaces, $p > 1$. The integral operator $K := B - A$ is not compact. For some physical conditions on K, the operator $(\lambda - A)^{-1}K$ or $K(\lambda - A)^{-1}$ are compact on L_p-spaces, $p > 1$. So, $\mathcal{K}(X) \subset_{\neq} \mathcal{G}_A(X)$ and $\mathcal{K}(X) \subset_{\neq} \mathcal{D}_A(X)$.

(d) For all $K \in \mathcal{G}_A(X)$, $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$.

(e) For all $K \in \mathcal{D}_A(X)$, $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$.

Proof of Theorem 2.1. We first claim that $\sigma_{\mathrm{ess}}(A) \subset \sigma_r(A)$ (respectively, $\sigma_{\mathrm{ess}}(A) \subset \sigma_l(A)$). Indeed, if $\lambda \notin \sigma_r(A)$ (respectively, $\lambda \notin \sigma_l(A)$) then there exists $K \in \mathcal{G}_A(X)$ (respectively, $K \in \mathcal{D}_A(x)$) such that $\lambda \in \rho(A + K)$; hence $\lambda \in \Phi_{(A+K)}$ and $i(\lambda - A - K) = 0$.

Let $\mu \in \rho(A)$; we have

$$(\lambda - A - K)^{-1}K = \big[I + (\lambda - A - K)^{-1}(\mu - \lambda + K)\big](\mu - A)^{-1}K \quad (2.1)$$

and

$$K(\lambda - A - K)^{-1} = K(\mu - A)^{-1}\big[I + (\mu - \lambda + K)(\lambda - A - K)^{-1}\big]. \quad (2.2)$$

Using (2.1) (respectively, (2.2)) and the fact that $\mathcal{I}(X)$ is a two-sided ideal of $\mathcal{L}(X)$, we infer that $(\lambda - A - K)^{-1}K \in \mathcal{I}(X)$ (respectively, $K(\lambda - A - K)^{-1} \in \mathcal{I}(X)$). Applying [26, Proposition 3.1(i)] we infer that $(I + (\lambda - A - K)^{-1}K)$ (respectively, $(I + K(\lambda - A - K)^{-1})$) is a Fredholm operator and $i(I + (\lambda - A - K)^{-1}K) = 0$ (respectively, $i(I + K(\lambda - A - K)^{-1}) = 0$). Using the equality $\lambda - A = (\lambda - A - K)(I + (\lambda - A - K)^{-1}K)$ (respectively, $\lambda - A = (I + K(\lambda - A - K)^{-1})(\lambda - A - K)$) together with Atkinson's theorem [31, p. 77, Proposition 2.c.7(ii)] one gets $\lambda \in \Phi_A$ and $i(\lambda - A) = 0$. Finally, the use of [40, p. 15, Theorem 4.5] shows that $\lambda \notin \sigma_{\mathrm{ess}}(A)$ which proves the claim.

On the other hand, since $\mathcal{K}(X) \subset \mathcal{G}_A(X)$ (respectively, $\mathcal{K}(X) \subset \mathcal{D}_A(X)$) we infer that $\sigma_r(A) \subset \sigma_{\mathrm{ess}}(A)$ (respectively, $\sigma_l(A) \subset \sigma_{\mathrm{ess}}(A)$) which completes the proof of theorem. □

By Lemma 4.1 of [21] and Theorem 2.1 we have:

Corollary 2.1. *Let X be a Banach space, $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the hypothesis* (H1) *and let $A \in \mathcal{C}(X)$. Then*

$$\sigma C(A) \subset \sigma_r(A) \quad \text{and} \quad \sigma R(A) \subset \sigma_r(A),$$
$$\sigma C(A) \subset \sigma_l(A) \quad \text{and} \quad \sigma R(A) \subset \sigma_l(A),$$

where $\sigma C(A)$ (*respectively, $\sigma R(A)$*) *denotes the continuous spectrum* (*respectively, the residual spectrum*) *of A.*

The following result provides a characterization of the right and left spectrums on a Banach space X.

Corollary 2.2. *Let X be a Banach space, $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the hypothesis* (H1) *and let $A \in \mathcal{C}(X)$. Then*

$$\lambda \notin \sigma_r(A) \quad \text{if and only if} \quad \lambda \in \Phi_A \quad \text{and} \quad i(\lambda - A) = 0,$$
$$\lambda \notin \sigma_l(A) \quad \text{if and only if} \quad \lambda \in \Phi_A \quad \text{and} \quad i(\lambda - A) = 0.$$

Proof. This corollary immediately follows from Theorem 2.1 and [40, p. 15, Theorem 4.5]. □

In practice (Section 3), we need the following corollary:

Corollary 2.3. *Let (Ω, Σ, μ) be an arbitrary positive measure space. If A is a closed densely defined linear operator on $L_p(\Omega)$ $p \geqslant 1$ then*

$$\sigma_{\mathrm{ess}}(A) = \bigcap_{C \in \mathcal{G}_A(L_p(\Omega))} \sigma(A + C) = \bigcap_{C \in \mathcal{D}_A(L_p(\Omega))} \sigma(A + C),$$

where $\mathcal{G}_A(L_p(\Omega)) = \{K \in \mathcal{L}(L_p(\Omega))$ such that $(\lambda - A)^{-1}K \in \mathcal{S}(L_p(\Omega))$ for some $\lambda \in \rho(A)\}$ and $\mathcal{D}_A(L_p(\Omega)) = \{K \in \mathcal{L}(L_p(\Omega))$ such that $K(\lambda - A)^{-1} \in \mathcal{S}(L_p(\Omega))$ for some $\lambda \in \rho(A)\}$.

Proof. $\mathcal{S}(L_p(\Omega))$ is a two-sided ideal of $\mathcal{L}(L_p(\Omega))$ satisfying the hypothesis (H1) (see [29, Proposition 3.4(iv)]). The result is now a consequence of Theorem 2.1. □

Remark 2.5. (a) For all $K \in \mathcal{G}_A(L_p(\Omega))$, $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$.
(b) For all $K \in \mathcal{D}_A(L_p(\Omega))$, $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$.

3. Application to transport equations

In this section we shall apply the results of Corollary 2.3 to the one-dimensional transport equation on L_p-spaces with $p \in [1, \infty)$.

Let

$$X_p = L_p\big[(-a,a)\times(-1,1); dx\,d\xi\big] \quad (a>0,\ 1\leqslant p<\infty)$$

and

$$\begin{aligned} X_p^o &:= L_p\big[\{-a\}\times(-1,0); |\xi|\,d\xi\big]\times L_p\big[\{a\}\times(0,1); |\xi|\,d\xi\big] \\ &:= X_{1,p}^o \times X_{2,p}^o \end{aligned}$$

equipped with the norm

$$\begin{aligned} \|\psi^o; X_p^o\| &= \big[\|\psi_1^o; X_{1,p}^o\|^p + \|\psi_2^o; X_{2,p}^o\|^p\big]^{1/p} \\ &= \left[\int_{-1}^{0} |\psi(-a,\xi)|^p|\xi|\,d\xi + \int_{0}^{1} |\psi(a,\xi)|^p|\xi|\,d\xi\right]^{1/p}. \end{aligned}$$

Moreover, we introduce

$$\begin{aligned} X_p^i &:= L_p\big[\{-a\}\times(0,1); |\xi|\,d\xi\big]\times L_p\big[\{a\}\times(-1,0); |\xi|\,d\xi\big] \\ &:= X_{1,p}^i \times X_{2,p}^i \end{aligned}$$

equipped with the norm

$$\begin{aligned} \|\psi^i; X_p^i\| &= \big[\|\psi_1^i; X_{1,p}^i\|^p + \|\psi_2^i; X_{2,p}^i\|^p\big]^{1/p} \\ &= \left[\int_{0}^{1} |\psi(-a,\xi)|^p|\xi|\,d\xi + \int_{-1}^{0} |\psi(a,\xi)|^p|\xi|\,d\xi\right]^{1/p}. \end{aligned}$$

We define the partial Sobolev space W_p by

$$W_p = \left\{\psi \in X_p \text{ such that } \xi\frac{\partial\psi}{\partial x} \in X_p\right\}.$$

It is well know that any function $\psi \in W_p$ has traces on $\{-a\}$ and $\{a\}$ in X_p^o and X_p^i (see, for instance, [2] or [8]). They are denoted, respectively, by ψ^o and ψ^i, and represent the outgoing and the incoming fluxes ("o" for outgoing and "i" for incoming).

We define the operator T_H by

$$\begin{cases} T_H : D(T_H) \subset X_p \to X_p, \\ \psi \to T_H\psi(x,\xi) = -\xi\frac{\partial\psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi), \\ D(T_H) = \{\psi \in W_p \text{ such that } H\psi^o = \psi^i\}, \end{cases}$$

where $\sigma(.) \in L^\infty(-1,1)$ and H is the boundary operator defined by

$$\begin{cases} H : X_p^o \to X_p^i, \\ H \in \mathcal{L}(X_p^o, X_p^i). \end{cases}$$

Note that the Schechter essential spectrum of the operator T_H was analyzed in [15]. In fact, if H is strictly singular operator then

$$\sigma_{\text{ess}}(T_H) = \left\{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\lambda^* := -\liminf_{|\xi|\to 0} \sigma(\xi)\right\}. \tag{3.1}$$

Next we consider the transport operator $A_H = T_H + K$, where K is the bounded operator given by

$$\begin{cases} K : X_p \to X_p, \\ \psi \to \int_{-1}^{1} \kappa(x,\xi,\xi')\psi(x,\xi')\,d\xi', \end{cases}$$

where $\kappa(.,.,.)$ is a measurable function from $[-a,a]\times[-1,1]\times[-1,1]$ to $\mathbf{R}$.

Observe that the operator K acts only on the variable ξ', so x may be viewed merely as a parameter in $[-a,a]$. Hence we may consider K as a function $K : x \in [-a,a] \to K(x) \in Z$, where $Z := \mathcal{L}(L_p([-1,1], d\xi))$.

In the following we will make the assumptions

$$(\mathrm{H2})\quad \begin{cases} K \text{ is measurable, i.e.,} \\ \quad \{x \in [-a,a] \text{ such that } K(x) \in \mathcal{O}\} \text{ is measurable} \\ \quad \text{if } \mathcal{O} \subset Z \text{ is open,} \\ \text{there exists a compact subset } T \subset Z \text{ such that} \\ \quad K(x) \in T \text{ a.e.,} \\ \text{and finally} \\ \quad K(x) \in \mathcal{K}(L_p([-1,1], d\xi)) \text{ a.e.,} \end{cases}$$

where $\mathcal{K}(L_p([-1,1], d\xi))$ denotes the set of all compact operators on $L_p([-1,1], d\xi)$.

Lemma 3.1 [15, Lemma 3.1]. *If K satisfies* (H2) *then, for any $\lambda \in \mathbf{C}$ such that* $\operatorname{Re}\lambda > -\lambda^*$, *the operator $(\lambda - T_H)^{-1}K$ is compact on X_p for $1 < p < \infty$ and weakly compact on X_1.*

Theorem 3.1. *Let $p \in [1,\infty)$. Suppose that the collision operator K satisfies the hypothesis* (H2) *on X_p and the boundary operator is strictly singular. Then*

$$\sigma_{\text{ess}}(A_H) = \sigma_{\text{ess}}(T_H) = \{\lambda \in \mathbf{C} \textit{ such that } \operatorname{Re}\lambda \leqslant -\lambda^*\}.$$

Remark 3.1. As we have already mentioned in Section 1, $\mathcal{W}(X_1) = \mathcal{S}(X_1)$. Accordingly, Theorem 3.1 is a natural extension to L_p-spaces ($1 < p < \infty$) of [27, Corollary 4.1 and Theorem 4.3]. Note also that, since $\mathcal{K}(X_p) \subset \mathcal{G}_{T_H}(X_p)$, Theorem 3.1 generalizes [21, Theorem 4.5 and Corollary 4.1].

Proof of Theorem 3.1. The hypothesis on K together with Lemma 3.1 implies that $K \in \mathcal{G}_{T_H}(X_p)$ for $\mathcal{I}(X_p) = \mathcal{S}(X_p)$. Now the result follows from Corollary 2.3, Remark 2.5(a) and Eq. (3.1). □

We consider the spectrum of the transport operator in a bounded convex geometry. Let $A_1\psi = B_1\psi + K_1\psi$, where

$$B_1\psi(r,v,\Omega) = -v\Omega \operatorname{grad}_r \psi(r,v,\Omega) - v\Sigma(r,v)\psi(r,v,\Omega),$$
$$K_1\psi(r,v,\Omega) = \int_D \int_E \kappa(r,v,\Omega,v',\Omega')\psi(r,v',\Omega')\,dv'\,d\Omega',$$

$D(A_1) = \{\psi \in L_p(G_1)\colon\ A_1\psi \in L_p(G_1)$ and $\psi(r,v,\Omega) = 0$ for $r \in \partial V$, Ω entering $V\}$, $D(B_1) = D(A_1)$, $D(K_1) = L_p(G_1)$ and $(r,v,\Omega) \in G_1 := V \times D \times E$, V is a bounded, closed convex domain in $\mathbf{R}^s$, $D = (0, v_m)$, $0 < v_m < \infty$, E is the surface of a unit sphere in $\mathbf{R}^s$, ∂V is the piecewise smooth surface of V. We assume that

(H3) $v\Sigma(r,v)$ and $\kappa(r,v,\Omega,v',\Omega')$ are nonnegative bounded measurable functions,

and set $\sigma_1 = \operatorname{ess\,inf}\{v\Sigma(r,v)\colon\ (r,v) \in V \times D\}$, $0 \leqslant \kappa(r,v,\Omega,v',\Omega') \leqslant k_1$, k_1 a positive constant. $v\Sigma(r,v)$ is a piecewise continuous function.

Similar to Ref. [20], we have $\sigma_{\mathrm{ess}}(B_1) = \sigma(B_1) = \{\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda \leqslant -\sigma_1\}$. For $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > -\sigma_1$, the operator $L_1(\lambda) := K_1(\lambda - B_1)^{-1}$ is compact on $L_p(G_1)$, $p > 1$, and weakly compact on $L_1(G_1)$ (see [44]). So, $L_1(\lambda) \in \mathcal{S}(L_p(G_1))$, $1 \leqslant p < \infty$, and we have $\sigma_{\mathrm{ess}}(B_1 + K_1) = \sigma_{\mathrm{ess}}(B_1)$ (see Remark 2.5(b)).

Theorem 3.2. *Under the hypothesis* (H3)*, we have*

$$\sigma_{\mathrm{ess}}(A_1) = \sigma_{\mathrm{ess}}(B_1) = \{\lambda \in \mathbf{C} \textit{ such that } \operatorname{Re}\lambda \leqslant -\sigma_1\}.$$

We consider the spectrum of the transport operator in a slab geometry with vacuum boundary conditions. Let $A_2\psi = B_2\psi + K_2\psi$, where

$$B_2\psi(x,v,\mu) = -v\mu\frac{\partial\psi}{\partial x}(x,v,\mu) - v\Sigma(x,v)\psi(x,v,\mu),$$
$$K_2\psi(x,v,\mu) = \int_{-1}^{1} \int_0^{v_m} \kappa(x,v,\mu,v',\mu')\psi(x,v',\mu')\,d\mu'\,dv',$$

$D(A_2) = \{\psi \in L_p(G_2)\colon\ A_2\psi \in L_p(G_2)$ and $\psi(-a,v,\mu) = \psi(a,v,-\mu) = 0$ for $0 < \mu \leqslant 1\}$, $D(B_2) = D(A_2)$, $D(K_2) = L_p(G_2)$ and $(x,v,\mu) \in G_2 := [-a,a] \times (0,v_m) \times [-1,1]$, $0 < v_m < \infty$, $L_p(G_2)$ is the Banach space of all complex functions $\psi(x,v,\mu)$ defined and pth power Lebesgue integrable on G_2, with the usual norm $\|\ \|_p$, $1 \leqslant p < \infty$. We assume that

(H4) $v\Sigma(x,v)$ and $\kappa(x,v,\mu,v',\mu')$ are nonnegative bounded measurable functions, and set $\sigma_2 = \operatorname{ess\,inf}\{v\Sigma(x,v)\colon (x,v) \in [-a,a] \times (0,v_m)\}$, $0 \leqslant \kappa(x,v,\mu,v',\mu') \leqslant k_2$, k_2 a positive constant. $v\Sigma(x,v)$ is a piecewise continuous function.

The Schechter essential spectrum $\sigma_{\text{ess}}(B_2) = \{\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda \leqslant -\sigma_2\}$. For $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > -\sigma_2$, the operator $L_2(\lambda) := K_2(\lambda - B_2)^{-1}$ is compact on $L_p(G_2)$, $p > 1$, and weakly compact on $L_1(G_2)$ (see [44]). So, $L_2(\lambda) \in \mathcal{S}(L_p(G_2))$, $1 \leqslant p < \infty$, and we have $\sigma_{\text{ess}}(B_2 + K_2) = \sigma_{\text{ess}}(B_2)$ (see Remark 2.5(b)).

Theorem 3.3. *Under the hypothesis* (H4), *we have*

$$\sigma_{\text{ess}}(A_2) = \sigma_{\text{ess}}(B_2) = \{\lambda \in \mathbf{C} \textit{ such that } \operatorname{Re}\lambda \leqslant -\sigma_2\}.$$

Next consider the spectrum of the transport operator in a slab geometry with generalized boundary conditions. Let $A_3\psi = B_3\psi + K_3\psi$, where

$$B_3\psi(x,v,\mu) = -v\mu\frac{\partial\psi}{\partial x}(x,v,\mu) - v\Sigma(x,v)\psi(x,v,\mu),$$

$$K_3\psi(x,v,\mu) = \int_{-1}^{1}\int_{0}^{v_m} \kappa(x,v,\mu,v',\mu')\psi(x,v',\mu')\,d\mu'\,dv',$$

$D(A_3) = \{\psi \in L_p(G_2)\colon\ A_3\psi \in L_p(G_2)$ and $\psi(a,v,-\mu) = \alpha(v,\mu)\psi(a,v,\mu)$, $\psi(-a,v,\mu) = \tau(v,\mu)\psi(-a,v,-\mu)$ for $0 < \mu \leqslant 1\}$, $D(B_3) = D(A_3)$, $D(K_3) = L_p(G_2)$. We assume that

(H5) $v\Sigma(x,v)$ and $\kappa(x,v,\mu,v',\mu')$ are nonnegative bounded measurable functions,

and set $\sigma_3 = \operatorname{ess\,inf}\{v\Sigma(x,v)\colon (x,v) \in [-a,a] \times (0,v_m)\}$, $0 \leqslant \kappa(x,v,\mu,v',\mu') \leqslant k_3$, k_3 a positive constant. $\alpha(v,\mu)$ and $\tau(v,\mu)$ are measurable functions satisfying $0 \leqslant \alpha(v,\mu) \leqslant 1$, $0 \leqslant \tau(v,\mu) \leqslant 1$, $\alpha(v,-\mu) = \alpha(v,\mu)$, $\tau(v,-\mu) = \tau(v,\mu)$.

The Schechter essential spectrum $\sigma_{\text{ess}}(B_3) = \{\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda \leqslant -\sigma_3\}$. For $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > -\sigma_3$, the operator $L_3(\lambda) := K_3(\lambda - B_3)^{-1}$ is compact on $L_p(G_2)$, $p > 1$, and weakly compact on $L_1(G_2)$ (see [44]). So, $L_3(\lambda) \in \mathcal{S}(L_p(G_2))$, $1 \leqslant p < \infty$, and we have $\sigma_{\text{ess}}(B_3 + K_3) = \sigma_{\text{ess}}(B_3)$ (see Remark 2.5(b)).

Theorem 3.4. *Under the hypothesis* (H5), *we have*

$$\sigma_{\text{ess}}(A_3) = \sigma_{\text{ess}}(B_3) = \{\lambda \in \mathbf{C} \textit{ such that } \operatorname{Re}\lambda \leqslant -\sigma_3\}.$$

We close this section by discussing briefly the Schechter essential spectrum of the transport operator in a slab geometry with generalized periodic boundary conditions. Let $A_4\psi = B_4\psi + K_4\psi$, where

$$B_4\psi(x,v,\mu) = -v\mu\frac{\partial\psi}{\partial x}(x,v,\mu) - v\Sigma(x,v)\psi(x,v,\mu),$$
$$K_4\psi(x,v,\mu) = \int_{-1}^{1}\int_{0}^{v_m}\kappa(x,v,\mu,v',\mu')\psi(x,v',\mu')\,d\mu'\,dv',$$

$D(A_4) = \{\psi \in L_p(G_2)\colon\ A_4\psi \in L_p(G_2)$ and $\psi(-a,v,\mu) = \alpha(v,\mu)\psi(a,v,\mu)$, $\psi(a,v,-\mu) = \tau(v,\mu)\psi(-a,v,-\mu)$ for $0 < \mu \leqslant 1\}$, $D(B_4) = D(A_4)$, $D(K_4) = L_p(G_2)$. We assume that

(H6) $v\Sigma(x,v)$ and $\kappa(x,v,\mu,v',\mu')$ are nonnegative bounded measurable functions,

and set $\sigma_4 = \operatorname{ess\,inf}\{v\Sigma(x,v)\colon\ (x,v) \in [-a,a]\times(0,v_m)\}$, $0 \leqslant \kappa(x,v,\mu,v',\mu') \leqslant k_4$, k_4 a positive constant. $\alpha(v,\mu)$ and $\tau(v,\mu)$ are measurable functions satisfying $0 \leqslant \alpha(v,\mu) \leqslant 1$, $0 \leqslant \tau(v,\mu) \leqslant 1$.

The Schechter essential spectrum $\sigma_{\mathrm{ess}}(B_4) = \{\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda \leqslant -\sigma_4\}$. For $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > -\sigma_4$, the operator $L_4(\lambda) := K_4(\lambda - B_4)^{-1}$ is compact on $L_p(G_2)$, $p > 1$, and weakly compact on $L_1(G_2)$ (see [44]). So, $L_4(\lambda) \in \mathcal{S}(L_p(G_2))$, $1 \leqslant p < \infty$, and we have $\sigma_{\mathrm{ess}}(B_4 + K_4) = \sigma_{\mathrm{ess}}(B_4)$ (see Remark 2.5(b)).

Theorem 3.5. *Under the hypothesis* (H6)*, we have*

$$\sigma_{\mathrm{ess}}(A_4) = \sigma_{\mathrm{ess}}(B_4) = \{\lambda \in \mathbf{C} \textit{ such that } \operatorname{Re}\lambda \leqslant -\sigma_4\}.$$

Remark 3.2. Note that Theorems 3.2, 3.3, 3.4 and 3.5 are, respectively, Theorems 1, 3, 5 and 7 in [44] which are based on Lemma D in [44]. But, the proof of this lemma is erroneous (see Remark 3.3 in [25]).

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